Forced harmonic and chaotic rotational oscillations

Recording and evaluating with CASSY

Experiment Objectives

- Recording a torsion pendulum's amplitude as a function of excitation frequency and damping.
- Determining the natural frequency of the torsion pendulum.
- Studying chaotic oscillations arising from the excitation of two equilibrium positions.

Fundamentals

Oscillations and waves are very important both in nature and in technology. The study of related phenomena therefore requires both the experimental and the theoretical angles. This helps understand the fundamental models and laws of physics.

Rotational oscillations represent a special case of mechanical oscillations. Yet all major phenomena can be studied on them.

This experiment studies forced rotational oscillations at different damping states.

Mounting additional weights on the torsion pendulum produces two equilibrium positions (or rest positions), which results in a chaotic behavior.

The physical value that fully describes the system's state at the given time t is the angle of deflection $\varphi(t)$ from the rest position (where $\varphi = 0$).

The spiral spring's effect on the torsion pendulum is given by Hooke's law:

$$M_f = D \cdot \varphi(t)$$

where D is the spring constant and M_f the torque on the torsion pendulum resulting from the spring.

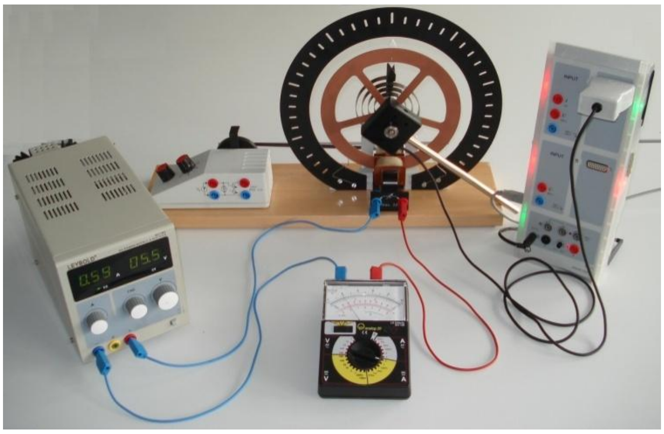

Fig. 1: Experiment setup for forced rotational oscillations.

In addition, the eddy-current brake exerts a torque on the pendulum:

$$M_r = k \cdot \dot{\varphi}(t)$$

where k is the constant of friction and $\dot{\varphi}(t)$ the first time derivative of the angle of deflection, so the angular velocity.

The sum of the two torques adds up to the total torque (since it is reversed):

$$-M_g = M_f + M_r$$

for which, per Newton:

$$M_g = J \cdot \ddot{\varphi}(t)$$

where J is the torsion pendulum's moment of inertia and $\ddot{\varphi}(t)$ the angular acceleration.

Hence:

$$J \cdot \ddot{\varphi}(t) + k \cdot \dot{\varphi}(t) + D \cdot \varphi(t) = 0 \,. \qquad (1)$$

Equation (1) is the equation of motion describing the free, damped oscillation. It is thereby a second-order ordinary, homogeneous, linear differential equation with a unique, known solution.

Introduce the following values to express the formulas more clearly:

- Damping coefficient

$$\delta = \frac{k}{2J}$$

- Natural frequency of the undamped torsion pendulum

$$\omega_0 = \sqrt{\frac{D}{J}}$$

- Frequency of the damped torsion pendulum

$$\omega = \sqrt{\omega_0^2 - \delta^2}$$

which exists only for $\omega_0 > \delta$.

YS 2013-08

Equipment

1 Torsion pendulum 346 00
1 DC Power Supply 0…16 V, 0…5 A 521 546
1 Plug-in power supply for torsion pendulum 562 793
2 Multimeter LDanalog 20 .... 531 120
1 Connecting lead, 100 cm, blue 500 442
2 Connecting leads, 100 cm, red/blue, pair 501 46
1 Rotary motion sensor S 524 082
1 Sensor-CASSY 2 .... 524 013
1 CASSY Lab 2 .... 524 220
1 PC with Windows XP/Vista/7/8

In this case, the general solution to equation (1) goes:

$$\varphi(t) = e^{-\delta t}\cdot(A\cdot \cos\omega t + B\cdot \sin\omega t).$$

Constants A and B are defined by the values specified for the launching angle $\varphi(0) = \varphi_0$ and the launching angle velocity $\dot{\varphi}(0) = \dot{\varphi}_0$, thus producing

$$A = \varphi_0 \text{ and } B = \left[\frac{\dot{\varphi}_0 + \varphi_0\cdot\delta}{\omega}\right]. \qquad (2)$$

Forced harmonic oscillations

If the torsion pendulum is stimulated externally by a periodic torque

$$M_{Ex} = M_0 \sin\omega_{Ex} t$$

therefore:

$$J\cdot\ddot{\varphi}(t) + k\cdot\dot{\varphi}(t) + D\cdot\varphi(t) = M_0\cdot\sin\omega_{ex}t. \qquad (3)$$

The solution to the inhomogeneous equation (3) is the sum of the homogeneous solution (2) and a special solution to (3). A special solution to (3) goes:

$$\varphi(t) = \varphi_0(\omega_{ex})\cdot\sin(\omega_{ex}t - \phi). \qquad (4)$$

Equation (4) inserted into (3) produces, after a few transformations for the amplitude:

$$\varphi_0(\omega_{ex}) = \frac{M_0}{J}\cdot\frac{1}{\sqrt{(\omega_0^2-\omega_{ex}^2)^2+\delta^2\cdot\omega_{ex}^2}}. \qquad (5)$$

Since the solution to the homogeneous equation tends to 0 when the time is high, equation (4) describes the angle of deflection $\varphi(t)$ after a settling time. The formula (5) applies for the torsion pendulum's amplitude (as a function of excitation frequency).

Note the amplitude tends to the fixed value M_0/I and not to 0 for small values of the excitation frequency. In contrast, the amplitude disappears when the excitations frequency is very large.

Differentiation (first derivative) of equation (5) for ω_{ex} yields a maximum value for the amplitude at the point

$$\omega_r = \sqrt{\omega_0^2 - 2\,\delta^2}.$$

This frequency is called the resonance frequency. Note at the limit $\delta \to 0$ the maximum amplitude tends to infinity, and:

$$\varphi_0(\omega_{ex}) = \frac{M_0}{J}\cdot\frac{1}{|\omega_0-\omega_{ex}|}. \qquad (6)$$

This phenomenon is designated as the resonance disaster.

For the phase shift ϕ in equation (4):

$$\phi = \arctan\left(2\cdot\frac{\delta\cdot\omega_{ex}}{\omega_0^2-\omega_{ex}^2}\right).$$

The phase shift disappears when the excitation frequency is very small.

The torsion pendulum vibrates with a displacement of 90° or π, opposite in phase, for very large values of ω_{ex}.
If the excitation frequency is equal to the natural frequency, so when $\omega_{ex} = \omega_0$ (case of resonance), then the phase shift is 45° or $\frac{\pi}{2}$. The excitation goes faster than the torsion pendulum.

Chaotic oscillations

The torsion pendulum's potential energy as a function of the displacement (of the angle of the deflection) is a quadratic function whose graph is a parabola with exactly one minimum at the point $\varphi = 0$. The torsion pendulum oscillates around this rest position (stable rest position, see the dotted curve in Fig. 2).

Mounting additional weights on the moving member yields two possible rest positions for the torsion pendulum – to the left and the right of the rest position without additional weights at $\varphi = 0$.

The potential energy is a function with two minima (stable rest positions) and a maximum between them (indifferent equilibrium or unstable rest position, see the continuous curve in Fig. 2).

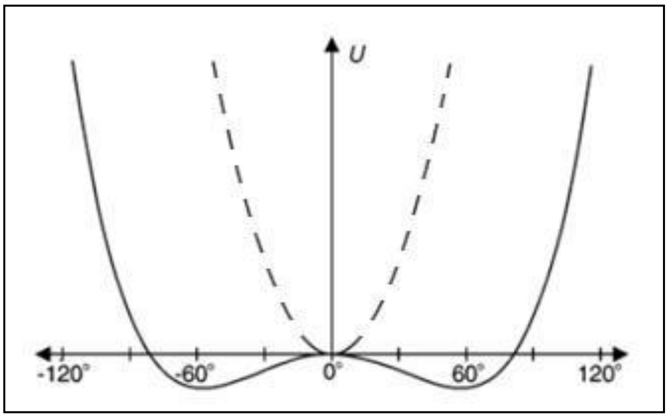

Fig. 2. The potential energy as a function of displacement.
Dotted curve: torsion pendulum without additional weights
Continuous curve: torsion pendulum with additional weights

The exciter supplies energy to the system according to the phase position. The moving member vibrates around one or the other rest position depending on the starting condition. This oscillation is not harmonic, at least for large amplitudes. A sufficiently large amplitude or energy overrides the maximum, and the moving member switches rest position around which to oscillate. This changes the phase position of the moving member's oscillation to the excitation vibration, and the moving member brakes or accelerates accordingly. Therefore, it now first vibrates around this rest position or immediately returns to it. The system behaves chaotically. This behavior is easy to observe and understand in this experiment.

Experiment setup

a) Forced harmonic oscillations

Fig. 1 represents the experiment's setup.

- Screw the stand rod into the rotary motion sensor.
- Carefully insert the shaft of the rotary motion sensor S into the provided socket on the pendulum (see Fig. 3, left). Do this without holding the pendulum, to avoid potential imbalance. The O-ring must be fully inserted on the shaft of the rotary motion sensor S for a nonslip connection of the two axes (see Fig. 3, right).

Fig. 3: Mounting of the rotary motion sensor S onto the torsion pendulum.

- Carefully lay the stand rod of the rotary motion sensor S on the desk (see Fig. 4) so the two axes are in a straight extension without.
  The weight of the stand rod holds the rotary motion sensor S stable without mechanical load on the moving member's shaft.

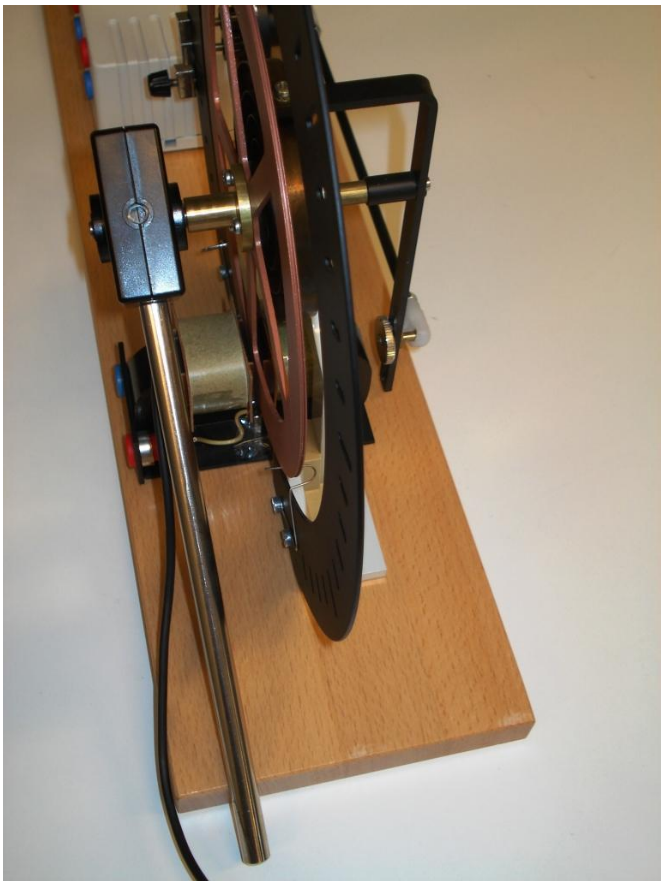

Fig. 4: The rotary motion sensor S on the torsion pendulum.

- Connect the rotary motion sensor S to the Sensor-CASSY 2.
- Connect the power supplies and the meters to the electric magnet for the eddy-current brake or to the excitation motor (Fig. 5).

Safety note

Pay attention to the maximum current on the electric magnet for the eddy-current brake:

I_{max} = 1 A (briefly 2 A)

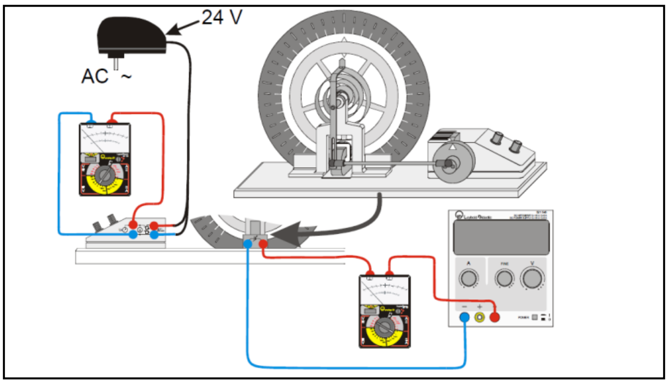

Fig. 5: Connection of the measuring instruments and power supplies.

- Do not switch the power supply for the eddy-current brake or the excitation motor on yet.

b) Chaotic oscillations

- For the measurement of chaotic oscillations, attach weights to the torsion pendulum's moving member right next to the displacement pointer. Make sure the weights are symmetrical around the pointer (Fig. 6).
- right next to the displacement pointer
- Slowly move the moving member slightly in one direction, and make sure a rest position is there. Repeat this process for the other side. If necessary, adjust the placement of the additional weights.

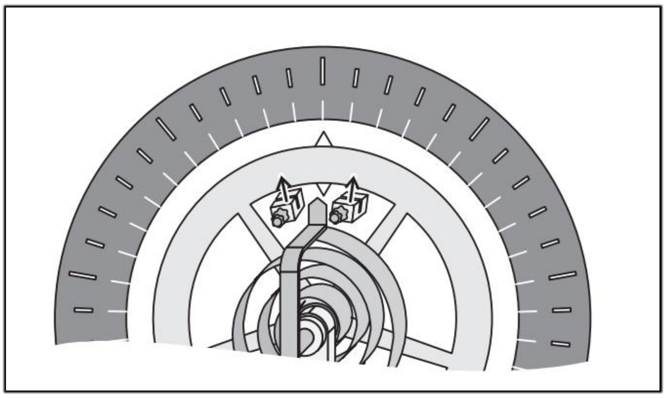

Fig. 6. Mounting of the additional weights to study chaotic oscillations.

Carrying out the Experiment

a) Forced harmonic oscillations

- Load the settings in CASSY Lab. Do not start the measurement yet!

With damping

- Set a low current (approx. 0.5 A) on the eddy-current brake.
- Switch the excitation motor on and observe the voltage applied. The voltage applied produces the excitation frequency ω_{ex}.
- The torsion pendulum starts oscillating. A constant amplitude appears after the settling time.

- As soon as the amplitude is constant, start the measurement in CASSY Lab with **Start measurement**. After recording the measurement value, immediately stop the measurement in CASSY Lab with . Only one measurement value is recorded.

- Repeat the experiment with different excitation frequencies.
- by slightly modifying the voltage on the excitation motor and waiting again for the settling time.
- Record the new measurement value with **Continue measurement**. After recording the measurement value, immediately stop the measurement in CASSY Lab with .

Note: Always record the measurement values using **Continue measurement** *and not* **Start measurement**.

Notes:
The measurement values recorded are on a curve as represented in Fig. 7.
If the measurement values concentrate in only a small area of the curve, continue the measurement for the missing values ω_{ex}.
Switch the power supply for the eddy-current brake off only when enough measurement values are recorded.

Without damping

- If necessary, switch the power supply for the eddy-current brake off.
- Repeat the measurements according to the description above.

Notes:
The amplitude at resonance frequency (ω_0) is so great the moving member hits the protective springs.
In this case, slightly change the voltage on the exciter.
Only record the measurement value if the moving member does not touch the protective springs.

b) Chaotic oscillations

Note:
This experiment does not require the eddy-current brake. If necessary, switch the power supply off, or do not connect it.

- Load the settings in CASSY Lab.
- Switch the excitation motor on and observe the oscillation.

Note:
Make sure the moving member does not oscillate only around one of the two rest positions, but it should switch from one rest position to the other. If necessary, change the excitation frequency using the voltage applied to the excitation motor.

- If the corresponding voltage is on, start the measurement in CASSY Lab with .

Note:
Record the oscillation over several minutes, so the chaotic nature of the oscillation is clear.

- Stop the measurement in CASSY Lab with .

Measurement Examples

a) Forced harmonic oscillations

With damping

Fig. 7 represents a measurement example of the damped forced harmonic oscillations. Equation (5) matches the measurement values (continuous curve):

$$\varphi_0(\omega_{ex}) = \frac{M_0}{J} \cdot \frac{1}{\sqrt{(\omega_0{}^2 - \omega_{ex}{}^2)^2 + \delta^2 \cdot \omega_{ex}{}^2}}.$$

Table 1 represents the corresponding current and fitting parameters.

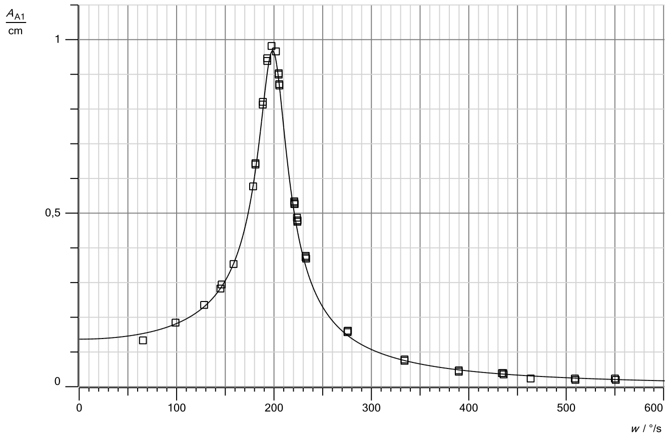

Fig. 7: Resonance curve of the damped forced harmonic oscillations.

Table 1: Fitting parameters for the damped forced oscillation

I	δ	ω_0	$\frac{M_0}{J}$
0.43 A	$28.29\ \frac{°}{s}$	$199.6\ \frac{°}{s}$	$5.45 \cdot 10^3\ \frac{1}{s^2}$

Without damping

Fig. 8 represents a measurement example of the undamped forced harmonic oscillations. The alignment of the measurement values follows equation (6)

$$\varphi_0(\omega_{ex}) = \frac{M_0}{J} \cdot \frac{1}{|\omega_0 - \omega_{ex}|}.$$

Table 2 represents the corresponding fitting parameters.

Table 2: Fitting parameters for the undamped forced oscillation

I	ω_0	$\frac{M_0}{J}$
0 A	$195.5\ \frac{°}{s}$	$5.27 \cdot 10^3\ \frac{1}{s^2}$

The "resonance disaster" occurs at the natural frequency.

The measurement values confirm the theoretical curves (5) and (6) very well.

Note: Instead of the angle of deflection, Fig.s 7 and 8 represent the displacement in cm proportional to the angle of deflection.

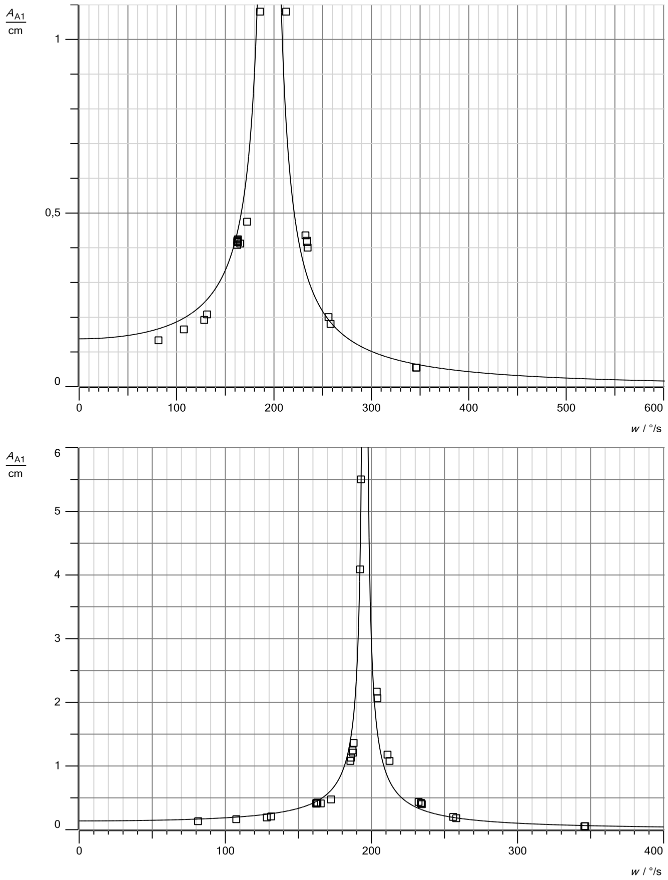

Fig. 8: Resonance curve of the undamped forced harmonic oscillations (different scaling of the axes: for direct comparison with the damped oscillation in the Fig. 7 graph above)

b) Chaotic oscillations

Fig.s 9 and 10 represent the angle of deflection as a function of time for a time interval of approx. 9 minutes and a 100-second section thereof.

There is no noticeable repeating pattern or period. It is therefore impossible to predict the displacement as a function of time. The system is not deterministic and behaves chaotically.

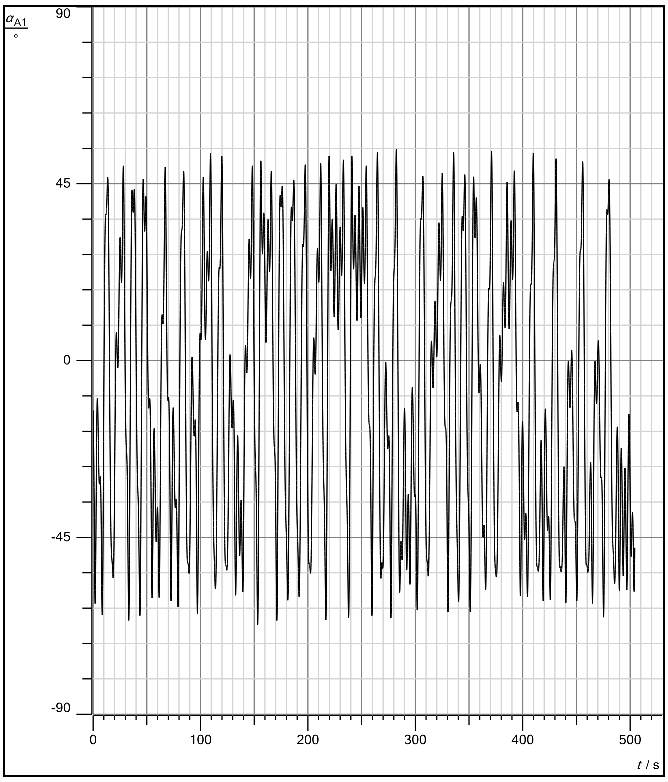

Fig. 9: Chaotic oscillations: recording time of 9 min.

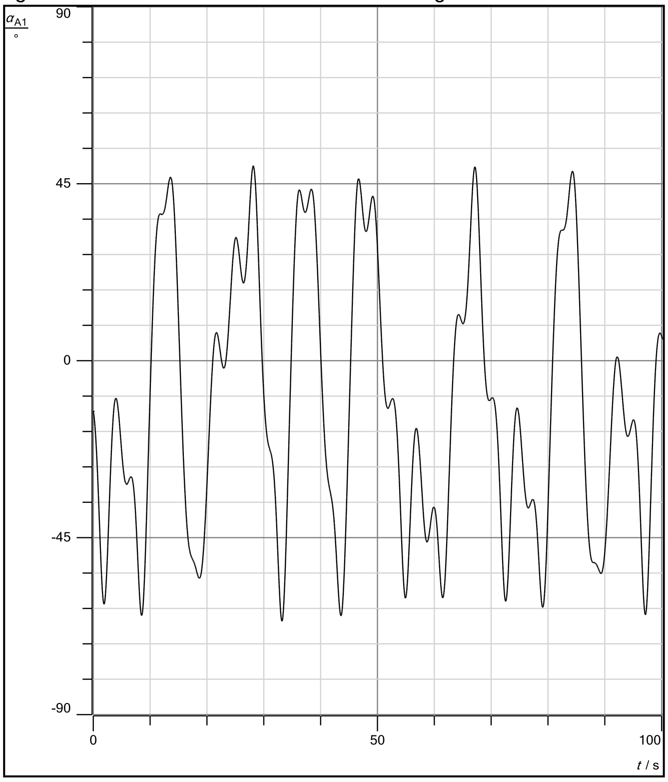

Fig. 10: Chaotic oscillations: recording time of 100 s.

An alternate representation of a physical system's dynamics is the phase diagram – e.g. the speed is plotted for location. In this case, the angular velocity (the first time derivative of the angle of deflection) is plotted against the angle of deflection. The resulting curves are called phase space trajectories.

Mathematically, we know deterministic solutions to the equation of motion (in the case of a unique solution, it is known for any length of time *t*) do not cross the phase space trajectories. Furthermore, these trajectories are closed curves in the case of periodic motion. Fig. 11 shows the phase diagram for the chaotic torsion pendulum over about 10 minutes. The spiral nature of the trajectories and the orbits of various points in the plane are clear characteristics of a chaotic motion.

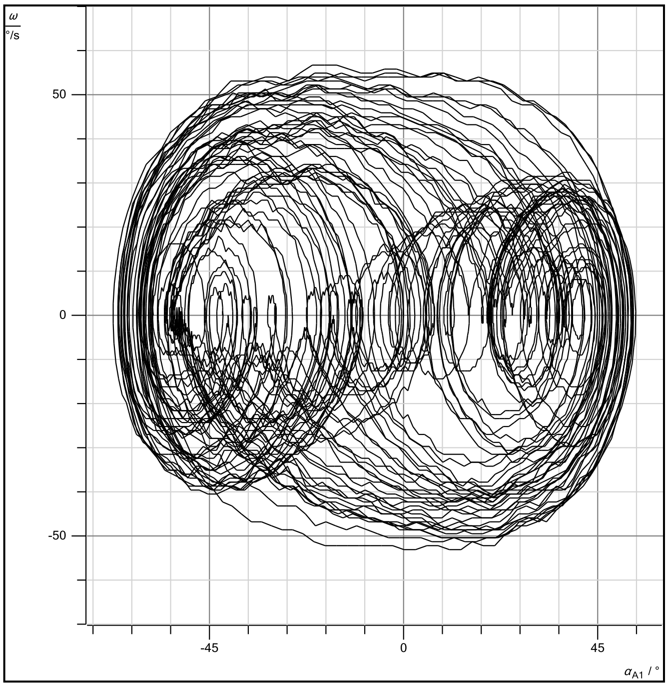

Fig. 11: Phase diagram of the chaotic oscillation. There are no visible closed trajectories.