Forced rotational oscillations

Measuring with a hand-held stop-clock

Objects of the experiment

■ Measuring the amplitude of forced rotational oscillations as function of exciter frequency for various damping constants

■ Determining the natural frequency of the oscillator

■ Investigating the phase shift between the exciter and the oscillator

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Principles

Oscillations (and wave) phenomena are well known due to their presence everywhere in nature and technique.

The rotary oscillations are a special case among various mechanical oscillator models which allow to investigate the most important phenomena occurring in all types of oscillations. In experiment P1.5.3.1 the free damped rotary oscillations have been investigated. In this experiment it will be investigated how the oscillator reacts to an external periodic force.

When applying the periodic torque

$$M_{ex} = M_0 \cdot \sin(\omega_{ex} \cdot t) \tag{I}$$

we obtain the following equation of motion for the damped rotary oscillating system (compare equation (I) in leaflet P1.5.3.1):

$$J\frac{d^2\varphi}{dt^2} + k\frac{d\varphi}{dt} + D \cdot \varphi = M_0 \cdot \sin(\omega_{ex} \cdot t) \tag{II}$$

J: moment of inertia

D: directional quantity (restoring torque)

k: damping coefficient (coefficient of friction)

φ: angle of rotation

M_0: maximum of external torque

$\omega_{ex} = 2\pi \cdot \nu$: frequency of the external torque

The solution of this inhomogeneous differential equation is the sum of a specific (particular) solution and the general solution of the corresponding homogeneous differential equation ($M_0 = 0$). The latter, however, decreases exponentially (compare equation (V) in leaflet P1.5.3.1) and is no longer significant after a sufficiently long period of time.
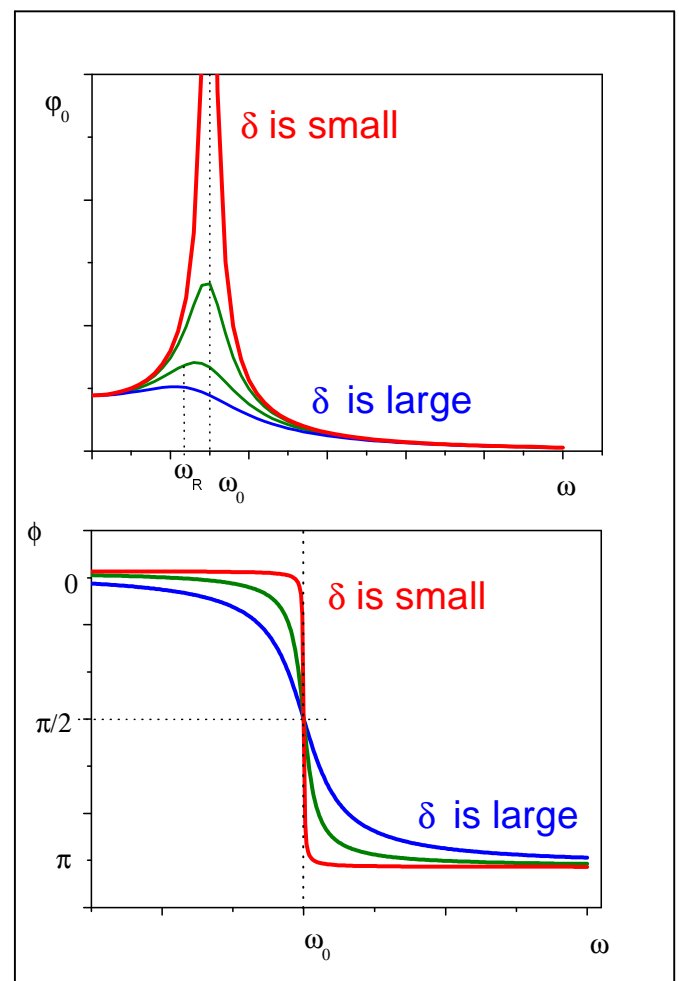

Fig. 1: Resonance curves (top) and phase shift between exciter and oscillator (bottom) for various damping constants δ.

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Apparatus

1 Torsion pendulum	346 00
1 DC power supply 0…16V/0…5 A	521 545
1 Plug-in power supply for torsion pendulum	562 793
1 Ammeter, DC, I ≤ 2 A, e.g. LDanalog 20	531 120
1 Voltmeter, DC, U ≤ 24 V, e.g. LDanalog 20	531 120
1 Connecting lead, 100 cm, blue	500 442
2 Pair cables, red and blue, 100 cm	501 46
1 Stop clock	313 07

For the specific solution the following relationship can be used:

$$\varphi(t) = \varphi_0(\omega_{ex}) \cdot \sin(\omega_{ex} \cdot t - \phi) \tag{III}$$

Substituting equation (III) in equation (II) gives after several trigonometric transformations the amplitude of the forced oscillation:

$$\varphi_0(\omega_{ex}) = \frac{M_0 / J}{\sqrt{(\omega_0 - \omega_{ex})^2 + \left(\dfrac{k}{J}\omega_{ex}\right)^2}} \tag{IV}$$

The frequency at which the amplitude of the oscillation is maximal is called the resonance frequency ω_R (amplitude resonance). This is the case when the radicand in the denominator is minimal. By equating the derivative of the radicand with respect to ω to zero the following relationship for the resonance frequency is found:

$$\omega_R = \sqrt{\omega_0^2 - \frac{k^2}{2 J^2}} = \sqrt{\omega_0^2 - 2\,\delta^2} \tag{VI}$$

with

$$\omega_0 = \sqrt{\frac{D}{J}} \qquad \text{(natural frequency)} \tag{VII}$$

$$\delta = \frac{k}{2 \cdot J} \qquad \text{(damping constant)} \tag{VIII}$$

The lower the damping the less the resonance frequency differs from the natural frequency ω_0 and the larger is the amplitude. In the limit of disappearing damping ($k \to 0$) the amplitude at the resonance frequency ($\omega_{ex} = \omega_0$) would tend towards infinity (so called resonance catastrophe).

From equation (IV) follows that amplitude of the forced oscillation tends towards zero for very high frequencies. For very low frequencies ($\omega \to 0$) the amplitude tends towards the value M_0/J (which is not equal zero). The resonance curve is not symmetrical with respect to the resonance frequency ω_R.

Fig. 2: Schematic representation (wiring diagram) of the experimental setup: (A) exciter, (B) eddy current brake.
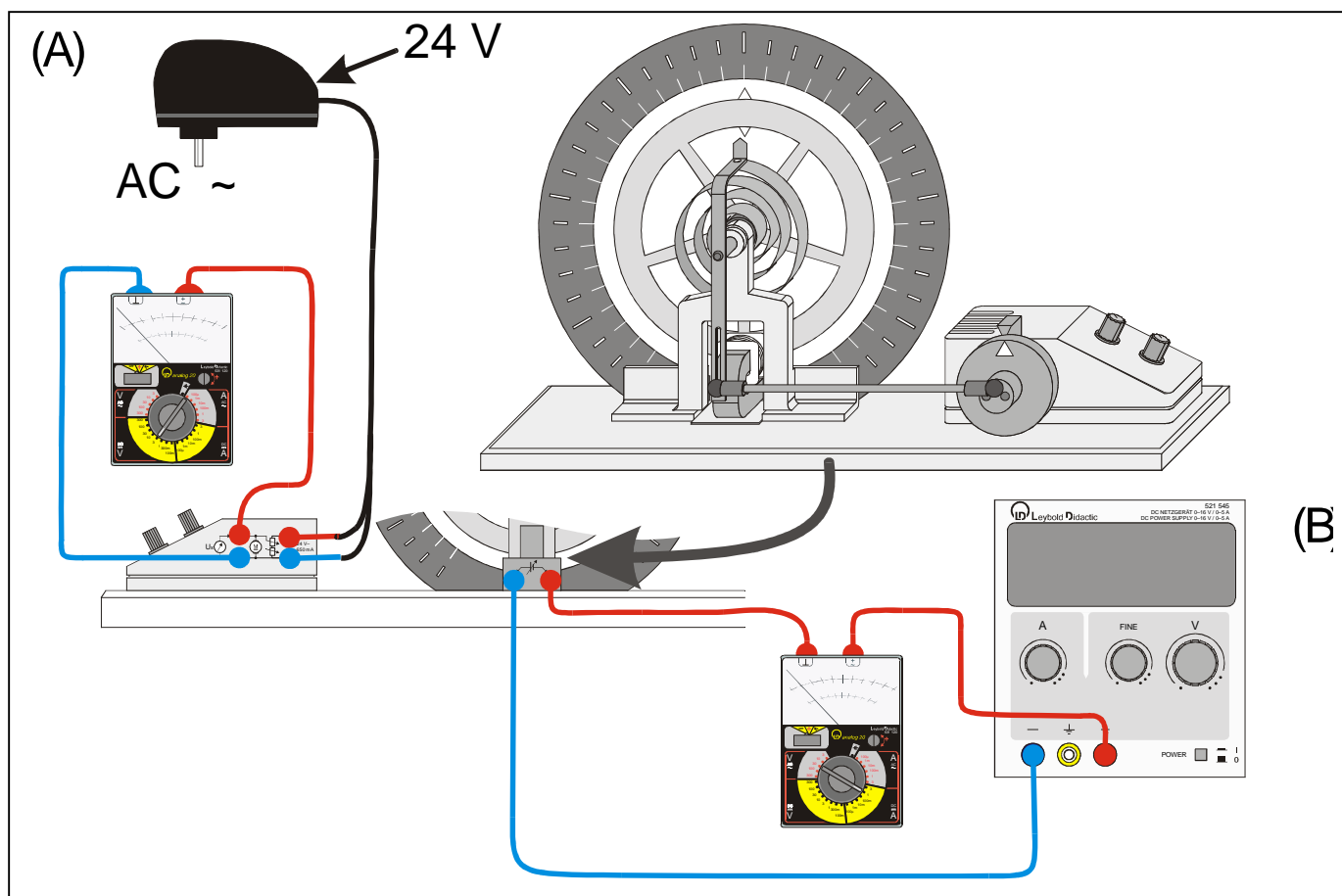
Note: The energy resonance has to be distinguished from the amplitude resonance considered above. It is possible to show that the oscillator possesses a maximum in energy when the frequency of the external torque equals the natural frequency: $\omega_{ex} = \omega_0$ (energy resonance). The energy and amplitude resonances are thus obtained at different excitation frequencies.

The phase shift ϕ between the external excitation and the oscillating system is given by:

$$\tan\phi = \frac{2\,\delta\,\omega_{ex}}{(\omega_0^2 - \omega_{ex}^2)} \qquad\qquad (IX)$$

From this relation follows:

For $\omega_{ex} \ll \omega_0$ the oscillator and the exciter oscillate almost in phase ($\phi \sim 0$).

For $\omega_{ex} \gg \omega_0$ the oscillator and the exciter oscillate almost in anti-phase ($\phi \sim \pi$).

For $\omega_{ex} = \omega_0$ the oscillator lags behind the exciter exactly by $\pi/2$.

Setup

The set up of the experiment is shown in Fig. 2 schematically. The period T of the exciter is measured by the stop clock (not shown in Fig. 2).

Safety notes

■ The current through the eddy current brake should not exceed 2 A for a long time.

■ Avoid overheating of the coils by measuring too long with large current I > 1 A.

Carrying out the experiment

a) Determining the amplitude as function of the frequency – recording the resonance curve

- Set the current for the electromagnet (eddy current brake) to a medium value, e.g. I = 0.4 A.
- Set the frequency of the exciter by adjusting the applied voltage – start with a small value, e.g. $\nu \sim 0.1$ Hz.

Hint: Measuring the applied voltage to the exciter serves in this experiment as a control only. For measuring the amplitude as function of the frequency ν of the exciter it is recommended to determine the period T of the exciter and evaluating the frequency $\nu = 1/T$ in turn while performing the experiment.

- Measure the period of the exciter and determine the frequency. To determine the period measure the time 10 T for 10 revolutions of the drive wheel.
- Read off the amplitude when the forced oscillation has reached a steady state and the amplitude of successive oscillations are constant.

Note: When measuring the amplitude as function of the frequency of the exciter, i.e. the resonance curve, several minutes have to be waited until the amplitude is sufficiently constant and the settling process of the forced oscillation has been completed. This holds especially for the case of weak damping. The settling process is particularly noticeable as a beat close to the resonance. (For this reason a medium current has to be chosen as starting value.)

- When changing the frequency of the exciter to a new value it might be necessary to readjust the voltage of the exciter after measuring and determining the frequency to have an appropriate frequency value with respect to the previous frequency setting.

  In the region of rapid amplitude increase the frequency has to be changed in small steps.

It is recommended to stop the oscillator completely between different exciter frequency settings and start the forced oscillation from scratch Thus the time of the settling process is minimized.

- Compare the motions of the pointers of the exciter and the oscillator. Observe the phase relation between exciter and oscillator qualitatively.

- Repeat the experiment for small (e.g. I = 0 A) and large damping (e.g. I = 0.7 A).

b) Determine the natural frequency of the oscillator

- Set the current of the eddy current brake to I = 0 A and deflect the pendulum to perform free rotary oscillations.

- Determine the natural frequency ν_0 by measuring 10-times the period T_0 for 10 oscillations with a deactivated eddy-current brake (I = 0 A).

- Calculate the natural frequency

$$\nu_0 = \frac{10}{T_0}$$

Note: The natural frequency ν_0 can be only estimated as a limit value of the almost undamped oscillation. Due to the inevitable frictional forces the oscillation is always damped.

Measuring example

Note: The measured experimental data may differ from pendulum to pendulum due to inevitable tolerances between various eddy current brakes and tolerances in the mechanical set up.

a) Determining the amplitude as function of the frequency – recording the resonance curve

Table. 1: 10-times the oscillation period T, frequency $\nu = 10/T$ and oscillation amplitude $A = \varphi_0$ for I = 0.4 A.

$\dfrac{10 \cdot T}{s}$	$\dfrac{\nu}{Hz}$	$\dfrac{A}{Scd}$
70.0	0.143	0.8
45.0	0.222	0.9
31.8	0.315	1.1
23.7	0.423	1.5
20.6	0.484	2.2
19.1	0.523	3.0
18.0	0.554	2.9
16.8	0.596	2.0
15.4	0.649	1.3
13.3	0.754	0.9
11.4	0.880	0.7
10.0	0.998	0.6

Table. 2: 10-times the oscillation period T, frequency $\nu = 10/T$ and oscillation amplitude $A = \varphi_0$ for I = 0 A.

$\dfrac{10 \cdot T}{s}$	$\dfrac{\nu}{Hz}$	$\dfrac{A}{Scd}$
39.9	0.250	1.0
30.1	0.332	1.1
30.0	0.333	1.1
25.1	0.399	1.4
20.9	0.479	2.5
19.5	0.514	4.8
18.7	0.535	9.6
18.2	0.551	15.4
17.9	0.560	18.7
17.8	0.563	19.4
17.4	0.574	7.2
17.4	0.576	3.8
17.2	0.581	3.4
15.9	0.629	1.7
13.1	0.762	0.9
11.3	0.889	0.7
10.0	0.997	0.6

Table. 3: 10-times the oscillation period T. frequency $\nu = 10/T$ and oscillation amplitude $A = \varphi_0$ for I = 0.79 A.

$\dfrac{10 \cdot T}{s}$	$\dfrac{\nu}{Hz}$	$\dfrac{A}{Scd}$
95.1	0.105	0.8
51.2	0.195	0.9
36.1	0.277	1.0
25.3	0.396	1.1
21.6	0.464	1.3
20.0	0.499	1.3
18.1	0.551	1.2
16.2	0.617	1.1
14.7	0.682	0.9
12.4	0.805	0.7
10.8	0.927	0.6
9.3	1.079	0.5

Phase:

- For small frequency values the pointer of the exciter and the oscillator are moving in phase.
- For large frequencies the phase between the pointer of the exciter and the oscillator is 180 deg.
- For large amplitudes, i.e. for frequencies near the resonance frequency, the phase displacement is approximately 90 deg.

b) Determine the natural frequency of the oscillator

The natural period measured over 10 oscillation periods have been determined to

$10 \cdot T_0 = 18$ s

from which the frequency is determined to:

$\nu_0 = 0.556$ Hz

Evaluation and results

Fig. 3 summarizes the results listed in the tables 1 to 3. The amplitude (A = φ_0) of the resonance curve decreases with increasing braking current. At the same time the peak of the resonance curve is shifted to lower frequency values in accordance with equation (VI). This is visualized in fig. 5 and is also confirmed by the fit results listed in Table 4.

For small frequency values the resonance cure tends towards a value which is different from zero as can be seen e.g. from Fig. 4 and Fig. 5. The resonance curve is not symmetrical with respect to the resonance frequency ν_R.

The resonance frequency ν_R (= $\omega/2\pi$) = 0.555 Hz for I = 0 A corresponds approximately to the natural frequency ν_0 = 0.556 Hz which was estimated by measuring 10-times the oscillation period of the free oscillation (i.e. almost undamped oscillation) with the deactivated eddy current brake.

The observed phase displacements are in accordance with equation (IX) and Fig. 1.

Table. 4: Parameter obtained by a fit of equation (IV) to the resonance curves plotted in Fig. 3. The fit parameters M_0 and k have been kept constant between the curves of various current settings.

$\dfrac{I}{A}$	$\dfrac{M_0}{N \, scd}$	$\dfrac{J}{kg \cdot scd^2}$	$\dfrac{\nu_0}{Hz}$	$\dfrac{k}{scd \cdot s^{-1}}$
0.0	0.62	3.0	0.555	$0.0084 \cdot 10^{-3}$
0.4	0.62	3.0	0.545	$0.38 \cdot 10^{-3}$
0.79	0.62	3.0	0.532	$0.89 \cdot 10^{-3}$

Fig. 4: Resonance curve for I = 0.4 A. The solid line is a guide to the eyes only.

Fig. 5: Shift of the resonance frequency for I = 0.4 A and I = 0.79 A. The solid lines are guides to the eyes only.
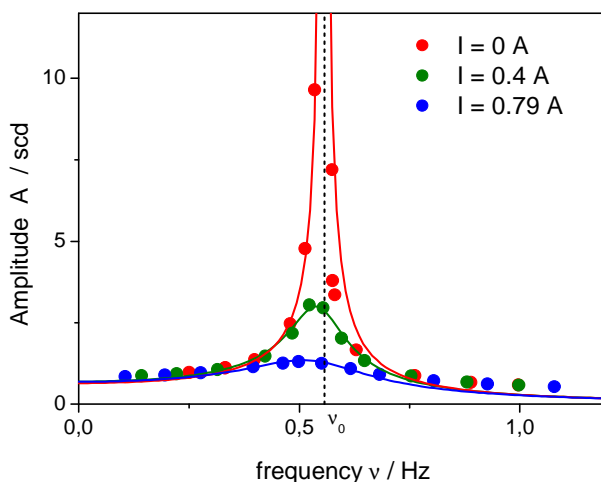

Fig. 3: Resonance curves for different braking currents. The solid lines correspond to fits according equation (IV). The fit parameters are listed in Table 4. ν_0 corresponds to the frequency for I = 0 A.
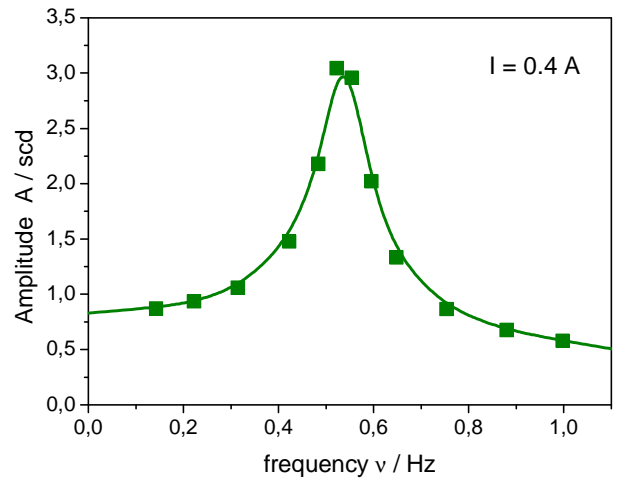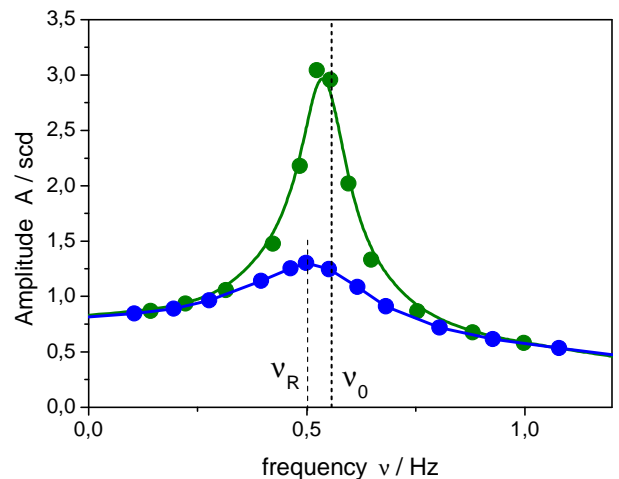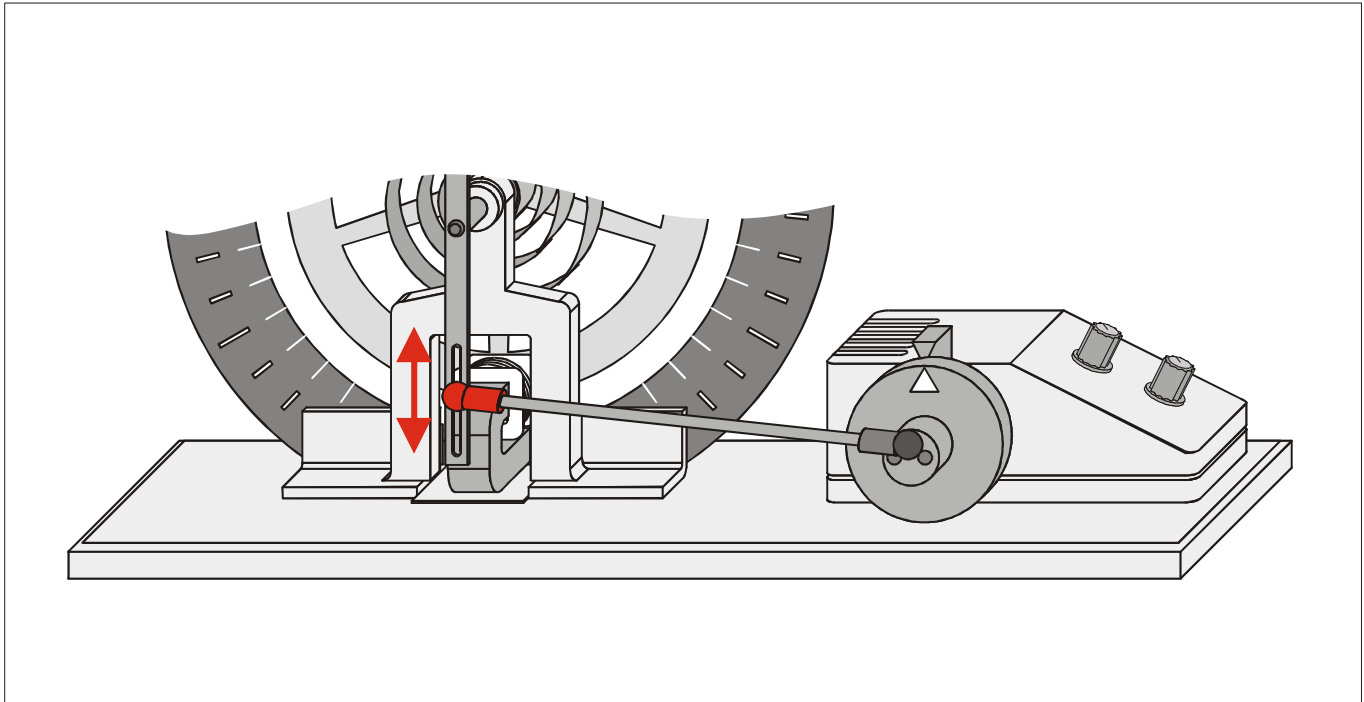
Fig. 6:   The connecting rod can be shifted to set the amplitude
of the exciter.

Supplementary information

All forced harmonic oscillations lead to a differential equation (II). The solution of this general oscillator model obtained in this experiment can thus be transferred other type of oscillations, e.g. forced electrical oscillations.

The influence of the exciter amplitude on the amplitude of the forced oscillation can be investigated by different adjustments of the connecting rod (Fig. 6).