Mechanics

Rotational motions of a rigid body
Moment of inertia

LD
Physics
Leaflets

P1.4.5.3

Steiner's theorem (parallel axis theorem)

Objects of the experiment

- Determining the moment of inertia of a circular disk for various distances between the axis of rotation and the axis of symmetry.
- Confirming *Steiner*'s theorem (parallel axis theorem).

Principles

The moment of inertia of an arbitrary rigid body whose mass elements Δm_i have the distances r_i from the axis of rotation A is

$$J_A = \sum_i \Delta m_i \cdot r_i^2 \qquad \text{(I).}$$

If the axis of rotation does not pass through the centre of mass of the body, application of Eq. (I) leads to an involved calculation. Often it is easier to calculate the moment of inertia J_S with respect to the axis S, which is parallel to the axis of rotation and passes through the centre of mass of the body.

For deriving the relation between J_A and J_S, the plane perpendicular to the axis of rotation where the respective mass element Δm_i is located is considered (see Fig. 1). In this plane, the vector $\boldsymbol{a}$ points from the axis of rotation to the centre-of-mass axis, the vector $\boldsymbol{r}_i$ points from the axis of rotation to the mass element Δm_i, and the vector $\boldsymbol{s}_i$ points from the centre-of-mass axis to the mass element. Thus

$$\boldsymbol{r}_i = \boldsymbol{a} + \boldsymbol{s}_i \qquad \text{(II),}$$

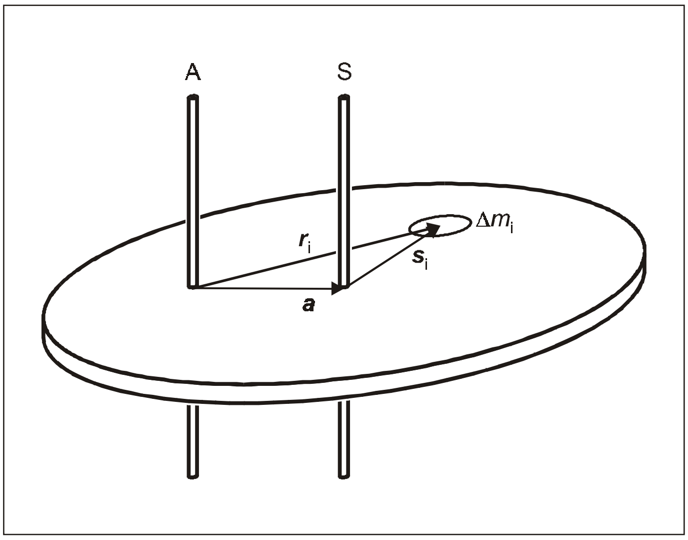

Fig. 1 Schematic illustration referring to the derivation of *Steiner*'s theorem (parallel axis theorem)

and the squares of the distances in Eq. (I) are

$$r_i^2 = (\boldsymbol{a} + \boldsymbol{s}_i)^2 = a^2 + 2 \cdot \boldsymbol{a} \cdot \boldsymbol{s}_i + s_i^2 \qquad \text{(III).}$$

Therefore the sum in Eq. (I) can be split into three terms:

$$J = \left(\sum_i \Delta m_i\right) \cdot a^2 + 2 \cdot \left(\sum_i \Delta m_i \cdot \boldsymbol{s}_i\right) \cdot \boldsymbol{a} + \sum_i \Delta m_i \cdot s_i^2 \qquad \text{(IV).}$$

In the first summand,

$$\sum_i \Delta m_i = M$$

is the total mass of the body. In the last summand,

$$\sum_i \Delta m_i \cdot s_i^2 = J_S$$

is the moment of inertia of the body with respect to the centre-of-mass axis. In the middle summand,

$$\sum_i \Delta m_i \cdot \boldsymbol{s}_i = 0$$

because the vectors $\boldsymbol{s}_i$ start from the axis through the centre of mass.

Thus *Steiner*'s theorem follows from Eq. (IV):

$$J_A = M \cdot a^2 + J_S \qquad \text{(V)}$$

This theorem will be verified in the experiment with a flat circular disk as an example. Its moment of inertia J_A with respect to an axis of rotation at a distance a from the axis of symmetry is obtained from the period of oscillation T of a torsion axle to which the circular disk is attached. We have

$$J_A = D \cdot \left(\frac{T}{2\pi}\right)^2 \qquad \text{(VI)}$$

D: restoring torque of the torsion axle

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Apparatus

1 torsion axle	347 80
1 circular disk for the torsion axle	347 83
1 stand base, V-shape, 20 cm	300 02
1 stopclock I, 30 s / 15 min	313 07

Setup and carrying out the experiment

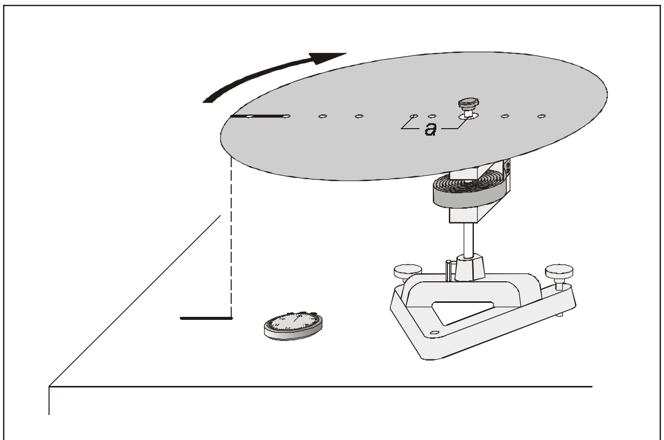
The experimental setup is illustrated in Fig 2.

- Fix the centre of the circular disk to the torsion axle and mark the equilibrium position on the table.
- Rotate the circular disk by 180° from the equilibrium position and release it.
- Start the time measurement as soon as the circular disk passes through the equilibrium position and stop the measurement after five oscillations.
- Repeat the measurement four times alternately deflecting the disk to the left and to the right.
- Calculate the period of oscillation *T* from the mean value of the five measurements.

- Mount the circular disk on the torsion axle so that its centre is at a distance of 2 cm from the axle, and, if necessary, mark the equilibrium position anew.
- Measure the time of five oscillations five times alternately deflecting the disk to the right and to the left.
- Calculate the period of oscillation *T*.

- Repeat the measurement for other distances *a* from the axis of symmetry.

Fig. 2 Experimental setup for the experimental confirmation of *Steiner*'s theorem (parallel axis theorem)



Measuring example

Mass of the disk: M = 704 g

Radius of the disk: R = 20 cm

Table 1: measured time of five oscillations for various distances *a* between the axis of rotation and the axis of symmetry and oscillation periods *T* calculated from the mean value of the measured values

$\frac{a}{\text{cm}}$	$\frac{5 \cdot T}{\text{s}}$					$\frac{T}{\text{s}}$
0	24.7	24.7	24.6	24.5	24.2	4.91
2	25.0	24.6	24.7	24.5	24.7	4.94
4	25.5	25.9	25.6	25.7	25.6	5.13
6	26.7	26.7	26.7	26.5	27.0	5.34
8	28.1	28.4	28.1	28.2	28.5	5.65
10	29.9	30.0	30.3	29.9	30.2	6.01
12	32.6	32.6	32.4	32.4	32.8	6.51
14	35.0	35.2	34.9	35.1	35.6	7.03
16	37.3	37.3	38.3	37.9	37.6	7.54

Evaluation

With the aid of Eq. (VI), the moment of inertia J_A can be calculated from the values of the period of oscillation listed in Table 1. The restoring torque *D* required for the calculation was determined in the experiment P1.4.5.1:

$$D = 0{,}023\ \frac{\text{Nm}}{\text{rad}}$$

The results are listed in Table 2.

Table 2: list of the squares a^2 of the distance *a* and the moments of inertia J_A calculated from the period of oscillation *T*

$\frac{a}{\text{cm}}$	$\frac{a^2}{\text{cm}^2}$	$\frac{\left(\frac{T}{2\pi}\right)^2}{\text{s}^2}$	$\frac{J_A}{\text{g} \cdot \text{m}^2}$
0	0	0.610	14.0
2	4	0.618	14.2
4	16	0.667	15.3
6	36	0.723	16.6
8	64	0.809	18.6
10	100	0.916	21.1
12	144	1.074	24.7
14	196	1.253	28.8
16	256	1.439	33.1

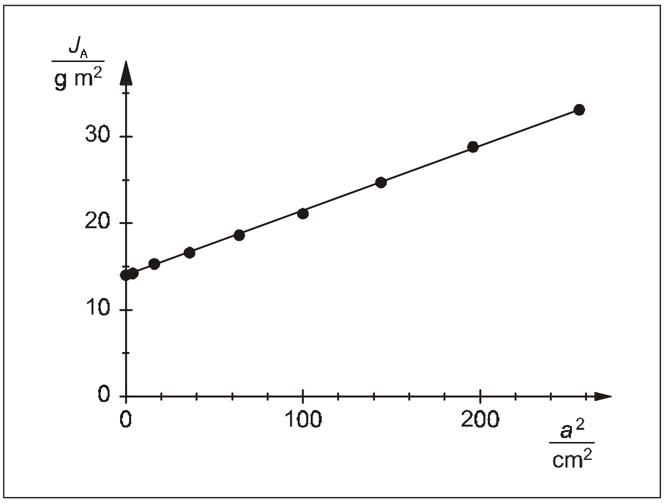


Fig. 3 Moment of inertia J_A as a function of the square of the distance *a* between the axis of rotation and the axis of symmetry

Eq. (V) describes a linear relation between J_A and a^2 with the slope *M* and the intercept of the ordinate J_S. This relation is confirmed by Fig. 3. The straight line drawn in the figure has the slope *M* = 740 g and the intercept of the ordinate J_s = 14 g m^2.

Result

The moment of inertia of a body with respect to an arbitrary axis can be calculated from the moment of inertia with respect to the centre-of-mass axis, the total mass and the distance between the two axes.